Inverse dynamics for a three-link planar chain

This text is part of an appendix in "Cesari, P., Shiratori, T., Olivato, P., Duarte, M. (2001) *Analysis of kinematically redundant reaching movements using the equilibrium-point hypothesis.* Biological Cybernetics, 84, 217-226." and is also included in the book "Zatsiorsky, VM (2002) *Kinetics of Human Motion*. Champaign, Human Kinetics".

The following convention applies to the notation used in this paper:

Subscript i runs 1, 2, or 3 meaning shoulder, elbow, or wrist joint when referring to angles, joint moments, or joint reaction forces; or meaning upper arm, forearm, or hand segment when referring to everything else.

x_i, y_i refer to the position of the center of mass of segment i in the horizontal or vertical direction, respectively

l_i is the length of segment i

d_i is the distance from the proximal joint of the segment i to its center of mass position

d_S is the distance from the wrist joint to the point of application of the spring force

m_i is the mass of segment i

I_i is the moment of inertia of segment i

F_{xi}, F_{yi} are the joint reaction forces of joint i in the horizontal or vertical direction, respectively

F_S is the spring force

T_i is the joint moment of joint i

g is the gravitational acceleration

Based on the model used here (see figure 1) the following relation applies to angles α, β, and θ:

$$\alpha_1=\beta_1=\theta_1 \quad (-)$$

$$\alpha_2=\theta_2 - \alpha_1 \qquad \beta_2=\pi-\alpha_2 \tag{1}$$

$\alpha_3=\theta_3 - \alpha_2 - \alpha_1$ $\qquad \beta_3=\pi-\alpha_3$

α_i and β_i are the angles in the "joint space", β_i is the internal angle of the joint i and α_I the external one; θ_i is the angle of the joint i in the "segment space".

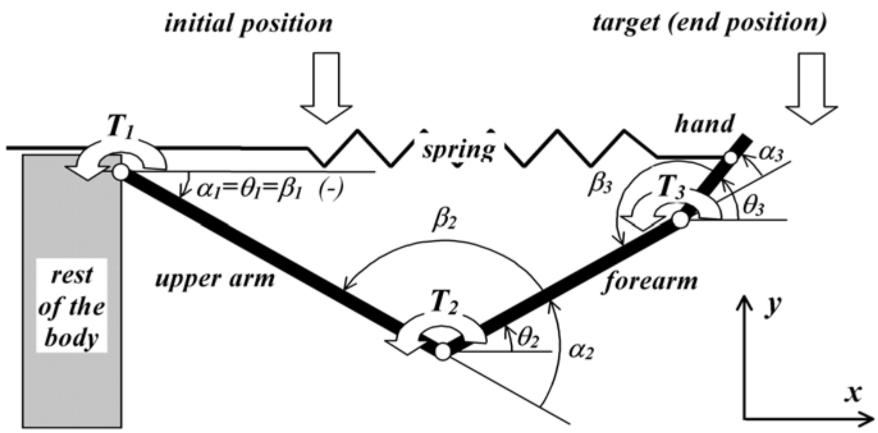

Figure 1. Model of the human body for the arm movement in the sagittal plane with the schematic location of the initial and target position. In this figure the spring force accounts for the external force on the hand (see the text).

In order to compute the equations of motion, the linear accelerations of the center of gravity of each link taking into account the constraints imposed by the kinematics of the linkage and starting from the shoulder joint as a fixed reference point were calculated by the first derivate of the Jacobian, *J*, of the respective angular velocities; or in a formal matrix form:

$$[\ddot{x}_i \;\; \ddot{y}_i\,]^{\mathrm{T}} = \dot{J}_i[\dot{\alpha}_1 \;\; \dot{\alpha}_2 \;\; \dot{\alpha}_3\,]^{\mathrm{T}} + J_i[\ddot{\alpha}_1 \;\; \ddot{\alpha}_2 \;\; \ddot{\alpha}_3\,]^{\mathrm{T}} \qquad i = 1...3 \tag{2}$$

The linear accelerations of the segment’s center of mass represented in figures 1 and 6 can be derived in a explicit form by:

Upper arm:

$$x_1 = d_1 \cos\alpha_1 \tag{3}$$

$$\ddot{x}_1 = -d_1(\ddot{\alpha}_1 \sin\alpha_1 + \dot{\alpha}_1^2 \cos\alpha_1) \tag{4}$$

$$y_1 = d_1 \sin\alpha_1 \tag{5}$$

$$\ddot{y}_1 = d_1(\ddot{\alpha}_1 \cos\alpha_1 - \dot{\alpha}_1^2 \sin\alpha_1) \tag{6}$$

Forearm:

$$x_2 = \ell_1 \cos\alpha_1 + d_2 \cos(\alpha_1 + \alpha_2) \tag{7}$$

$$\begin{aligned} \ddot{x}_2 = &-[\ell_1 \sin\alpha_1 + d_2 \sin(\alpha_1 + \alpha_2)]\ddot{\alpha}_1 - d_2 \sin(\alpha_1 + \alpha_2)\ddot{\alpha}_2 \\ &-[\ell_1 \cos\alpha_1 + d_2 \cos(\alpha_1 + \alpha_2)]\dot{\alpha}_1^2 - d_2 \cos(\alpha_1 + \alpha_2)(2\dot{\alpha}_1\dot{\alpha}_2 + \dot{\alpha}_2^2) \end{aligned} \tag{8}$$

$$y_2 = \ell_1 \sin\alpha_1 + d_2 \sin(\alpha_1 + \alpha_2) \tag{9}$$

$$\begin{aligned} \ddot{y}_2 = &[\ell_1 \cos\alpha_1 + d_2 \cos(\alpha_1 + \alpha_2)]\ddot{\alpha}_1 + d_2 \cos(\alpha_1 + \alpha_2)\ddot{\alpha}_2 \\ &-[\ell_1 \sin\alpha_1 - d_2 \sin(\alpha_1 + \alpha_2)]\dot{\alpha}_1^2 + d_2 \sin(\alpha_1 + \alpha_2)(2\dot{\alpha}_1\dot{\alpha}_2 + \dot{\alpha}_2^2) \end{aligned} \tag{10}$$

Hand:

$$x_3 = \ell_1 \cos\alpha_1 + \ell_2 \cos(\alpha_1 + \alpha_2) + d_3 \cos(\alpha_1 + \alpha_2 + \alpha_3) \tag{11}$$

$$\begin{aligned} \ddot{x}_3 = &-[\ell_1 \sin\alpha_1 + \ell_2 \sin(\alpha_1 + \alpha_2) + d_3 \sin(\alpha_1 + \alpha_2 + \alpha_3)]\ddot{\alpha}_1 \\ &-[\ell_2 \sin(\alpha_1 + \alpha_2) + d_3 \sin(\alpha_1 + \alpha_2 + \alpha_3)]\ddot{\alpha}_2 - d_3 \sin(\alpha_1 + \alpha_2 + \alpha_3)\ddot{\alpha}_3 \\ &-[\ell_1 \cos\alpha_1 + \ell_2 \cos(\alpha_1 + \alpha_2) + d_3 \cos(\alpha_1 + \alpha_2 + \alpha_3)]\dot{\alpha}_1^2 \\ &-[\ell_2 \cos(\alpha_1 + \alpha_2) + d_3 \cos(\alpha_1 + \alpha_2 + \alpha_3)]\dot{\alpha}_2^2 - d_3 \cos(\alpha_1 + \alpha_2 + \alpha_3)\dot{\alpha}_3^2 \\ &-[\ell_2 \cos(\alpha_1 + \alpha_2) + d_3 \cos(\alpha_1 + \alpha_2 + \alpha_3)]2\dot{\alpha}_1\dot{\alpha}_2 \\ &-d_3 \cos(\alpha_1 + \alpha_2 + \alpha_3)(2\dot{\alpha}_1\dot{\alpha}_3 + 2\dot{\alpha}_2\dot{\alpha}_3) \end{aligned} \tag{12}$$

$$y_3 = \ell_1 \sin\alpha_1 + \ell_2 \sin(\alpha_1 + \alpha_2) + d_3 \sin(\alpha_1 + \alpha_2 + \alpha_3) \tag{13}$$

$$\begin{aligned} \ddot{y}_3 = &[\ell_1 \cos\alpha_1 + \ell_2 \cos(\alpha_1 + \alpha_2) + d_3 \cos(\alpha_1 + \alpha_2 + \alpha_3)]\ddot{\alpha}_1 \\ &+[\ell_2 \cos(\alpha_1 + \alpha_2) + d_3 \cos(\alpha_1 + \alpha_2 + \alpha_3)]\ddot{\alpha}_2 + d_3 \cos(\alpha_1 + \alpha_2 + \alpha_3)\ddot{\alpha}_3 \\ &-[\ell_1 \sin\alpha_1 + \ell_2 \sin(\alpha_1 + \alpha_2) + d_3 \sin(\alpha_1 + \alpha_2 + \alpha_3)]\dot{\alpha}_1^2 \\ &-[\ell_2 \sin(\alpha_1 + \alpha_2) + d_3 \sin(\alpha_1 + \alpha_2 + \alpha_3)]\dot{\alpha}_2^2 - d_3 \sin(\alpha_1 + \alpha_2 + \alpha_3)\dot{\alpha}_3^2 \\ &-[\ell_2 \sin(\alpha_1 + \alpha_2) + d_3 \cos(\alpha_1 + \alpha_2 + \alpha_3)]2\dot{\alpha}_1\dot{\alpha}_2 \\ &-d_3 \sin(\alpha_1 + \alpha_2 + \alpha_3)(2\dot{\alpha}_1\dot{\alpha}_3 + 2\dot{\alpha}_2\dot{\alpha}_3) \end{aligned} \tag{14}$$

Based on the free body diagrams, the equations of motion in the sagittal plane were derived by means of the Newton-Euler method.

Hand:

$$m_3\ddot{x}_3 = F_{x3} - F_S \qquad (15)$$

$$m_3\ddot{y}_3 = F_{y3} - m_3 g \qquad (16)$$

$$\begin{aligned} I_3\ddot{\alpha}_3 = {} & T_3 + F_{x3}d_3\sin(\alpha_1+\alpha_2+\alpha_3) - F_{y3}d_3\cos(\alpha_1+\alpha_2+\alpha_3) \\ & + F_S d_S \sin(\alpha_1+\alpha_2+\alpha_3) - I_3\ddot{\alpha}_1 - I_3\ddot{\alpha}_2 \end{aligned} \qquad (17)$$

Forearm:

$$m_2\ddot{x}_2 = F_{x2} - F_{x3} \qquad (18)$$

$$m_2\ddot{y}_2 = F_{y2} - F_{y3} - m_2 g \qquad (19)$$

$$\begin{aligned} I_2\ddot{\alpha}_2 = {} & T_2 - T_3 + F_{x3}(\ell_2 - d_2)\sin(\alpha_1+\alpha_2) - F_{y3}(\ell_2 - d_2)\cos(\alpha_1+\alpha_2) \\ & + F_{x2}d_2\sin(\alpha_1+\alpha_2) - F_{y2}d_2\cos(\alpha_1+\alpha_2) - I_2\ddot{\alpha}_1 \end{aligned} \qquad (20)$$

Upper arm:

$$m_1\ddot{x}_1 = F_{x1} - F_{x2} \qquad (21)$$

$$m_1\ddot{y}_1 = F_{y1} - F_{y2} - m_1 g \qquad (22)$$

$$\begin{aligned} I_1\ddot{\alpha}_1 = {} & T_1 - T_2 + F_{x2}(\ell_1 - d_1)\sin\alpha_1 - F_{y2}(\ell_1 - d_1)\cos\alpha_1 \\ & + F_{x1}d_1\sin\alpha_1 - F_{y1}d_1\cos\alpha_1 \end{aligned} \qquad (23)$$

The joint moments can be straightforwardly found solving the set of equations (15-23) in a top-down way (first the wrist moment and so on) once the inertial parameters and the kinematics data are known (the accelerations in the set of equations (3-14) are found first). If the kinematics data are time-series, the equations are implemented in any programming package and the time-series of the joint moments are obtained. In this paper all data processing was implemented in Matlab. In a matrix-vector form (useful for viasualizing the moment components, coupling and transfer effects) the joint moments can be represented as:

$$T = M(\alpha)\ddot{\alpha} + v(\alpha,\dot{\alpha}) + G(\alpha) + T_{ext}(\alpha) \qquad (24)$$

or

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} M(\alpha)_{1,1} & M(\alpha)_{1,2} & M(\alpha)_{1,3} \\ M(\alpha)_{2,1} & M(\alpha)_{2,2} & M(\alpha)_{2,3} \\ M(\alpha)_{3,1} & M(\alpha)_{3,2} & M(\alpha)_{3,3} \end{bmatrix} \begin{bmatrix} \ddot{\alpha}_1 \\ \ddot{\alpha}_2 \\ \ddot{\alpha}_3 \end{bmatrix} + \begin{bmatrix} v(\alpha,\dot{\alpha})_1 \\ v(\alpha,\dot{\alpha})_2 \\ v(\alpha,\dot{\alpha})_3 \end{bmatrix} + \begin{bmatrix} G(\alpha)_1 \\ G(\alpha)_2 \\ G(\alpha)_3 \end{bmatrix} + \begin{bmatrix} T_{ext}(\alpha)_1 \\ T_{ext}(\alpha)_2 \\ T_{ext}(\alpha)_3 \end{bmatrix} \quad (24a)$$

Where:

T is the vector of joint moments (3x1).

$M(\alpha)$ is the inertia matrix (3x3). Because inertia matrices are symmetric ($I_{1,2}$=$I_{2,1}$, $I_{1,3}$=$I_{3,1}$ and $I_{2,3}$=$I_{3,2}$), only six elements of the matrix should be determined. The elements are the three moments of inertia, $I_{1,1}$, $I_{2,2}$ and $I_{3,3}$

$\ddot{\alpha}$ is the vector of angular accelerations (3x1).

$v(\alpha,\dot{\alpha})$ is the vector of centrifugal/Coriolis terms (3x1).

$G(\alpha)$ is the vector of gravity terms (3x1).

T_{ext} is the vector of joint moments due to other external forces besides gravity; in this case, represents the moment due to the spring force (3x1).

The motion equations were rearranjed in the above format and the correspondent terms are:

$$T = [T_1 \;\; T_2 \;\; T_3]^{\mathrm{T}} \quad (25)$$

$$\ddot{\alpha} = [\ddot{\alpha}_1 \;\; \ddot{\alpha}_2 \;\; \ddot{\alpha}_3]^{\mathrm{T}} \quad (26)$$

$$\begin{aligned} M(\alpha)_{1,1} &= m_1 d_1^2 + I_1 + m_2(\ell_1^2 + d_2^2 + 2\ell_1 d_2 \cos\alpha_2) + I_2 \\ &+ m_3[\ell_1^2 + \ell_2^2 + d_3^2 + 2\ell_1\ell_2\cos\alpha_2 + 2\ell_1 d_3 \cos(\alpha_2+\alpha_3) + 2\ell_2 d_3 \cos\alpha_3] + I_3 \end{aligned} \quad (27)$$

$$\begin{aligned} M(\alpha)_{1,2} &= m_2(d_2^2 + \ell_1 d_2 \cos\alpha_2) + I_2 + m_3[\ell_2^2 + d_3^2 + \ell_1\ell_2\cos\alpha_2 \\ &+ \ell_1 d_3 \cos(\alpha_2+\alpha_3) + 2\ell_2 d_3 \cos\alpha_3] + I_3 \end{aligned} \quad (28)$$

$$M(\alpha)_{1,3} = m_3[d_3^2 + \ell_1 d_3 \cos(\alpha_2+\alpha_3) + \ell_2 d_3 \cos\alpha_3] + I_3 \quad (29)$$

$$\begin{aligned} M(\alpha)_{2,1} &= m_2(d_2^2 + \ell_1 d_2 \cos\alpha_2) + I_2 \\ &+ m_3[\ell_2^2 + d_3^2 + \ell_1\ell_2\cos\alpha_2 + 2\ell_2 d_3\cos\alpha_3 + \ell_1 d_3 \cos(\alpha_2+\alpha_3)] + I_3 \end{aligned} \quad (30)$$

$$M(\alpha)_{2,2} = m_2 d_2^2 + I_2 + m_3(\ell_2^2 + d_3^2 + \ell_2 d_3 \cos\alpha_3) + I_3 \tag{31}$$

$$M(\alpha)_{2,3} = m_3(d_3^2 + \ell_2 d_3 \cos\alpha_3) + I_3 \tag{32}$$

$$M(\alpha)_{3,1} = m_3[d_3^2 + \ell_1 d_3 \cos(\alpha_2 + \alpha_3) + \ell_2 d_3 \cos\alpha_3] + I_3 \tag{33}$$

$$M(\alpha)_{3,2} = m_3(d_3^2 + \ell_2 d_3 \cos\alpha_3) + I_3 \tag{34}$$

$$M(\alpha)_{3,3} = m_3 d_3^2 + I_3 \tag{35}$$

$$\begin{aligned} v(\alpha,\dot\alpha)_1 = &-[(m_2\ell_1 d_2 + m_3\ell_1\ell_2)\sin\alpha_2 + m_3\ell_1 d_3\sin(\alpha_2+\alpha_3)](2\dot\alpha_1\dot\alpha_2 + \dot\alpha_2^2) \\ &-[m_3\ell_1 d_3\sin(\alpha_2+\alpha_3) + m_3\ell_2 d_3\sin\alpha_3](2\dot\alpha_1\dot\alpha_3 + 2\dot\alpha_2\dot\alpha_3 + \dot\alpha_3^2) \end{aligned} \tag{36}$$

$$\begin{aligned} v(\alpha,\dot\alpha)_2 = &[(m_3\ell_1\ell_2 + m_2 d_2\ell_1)\sin\alpha_2 + m_3 d_3\ell_1\sin(\alpha_2+\alpha_3)]\dot\alpha_1^2 \\ &- m_3 d_3\ell_2\sin\alpha_3(2\dot\alpha_1\dot\alpha_3 + 2\dot\alpha_2\dot\alpha_3 + \dot\alpha_3^2) \end{aligned} \tag{37}$$

$$\begin{aligned} v(\alpha,\dot\alpha)_3 = &[m_3\ell_1 d_3\sin(\alpha_2+\alpha_3) + m_3\ell_2 d_3\sin\alpha_3]\dot\alpha_1^2 \\ &+ m_3\ell_2 d_3\sin\alpha_3(2\dot\alpha_1\dot\alpha_2 + \dot\alpha_2^2) \end{aligned} \tag{38}$$

$$\begin{aligned} G(\alpha)_1 = &m_1 g d_1\cos\alpha_1 + m_2 g[\ell_1\cos\alpha_1 + d_2\cos(\alpha_1+\alpha_2)] \\ &+ m_3 g[\ell_1\cos\alpha_1 + \ell_2\cos(\alpha_1+\alpha_2) + d_3\cos(\alpha_1+\alpha_2+\alpha_3)] \end{aligned} \tag{39}$$

$$G(\alpha)_2 = m_2 g d_2\cos(\alpha_1+\alpha_2) + m_3 g[\ell_2\cos(\alpha_1+\alpha_2) + d_3\cos(\alpha_1+\alpha_2+\alpha_3)] \tag{40}$$

$$G(\alpha)_3 = m_3 g d_3\cos(\alpha_1+\alpha_2+\alpha_3) \tag{41}$$

$$T_{ext}(\alpha)_1 = -F_S\ell_1\sin\alpha_1 - F_S\ell_2\sin(\alpha_1+\alpha_2) - F_S(d_3+d_S)\sin(\alpha_1+\alpha_2+\alpha_3) \tag{42}$$

$$T_{ext}(\alpha)_2 = -F_S\ell_2\sin(\alpha_1+\alpha_2) - F_S(d_3+d_S)\sin(\alpha_1+\alpha_2+\alpha_3) \tag{43}$$

$$T_{ext}(\alpha)_3 = -F_S(d_3+d_S)\sin(\alpha_1+\alpha_2+\alpha_3) \tag{44}$$